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SEIR Mathematical Model with Treatment at The Latent Class

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ABSTRACT

In this paper, we discussed a tuberculosis disease using deterministic model to study the control of the disease. We also discussed the stability of the given SEIR model. In this, we see that the exposed individual is very important in controlling the disease. We find that if more and more people at exposed class go for treatment during this state, the disease will not spread quickly with time and we can implement some more preventive measures to control it.

Keywords: *SIR Model, SEIR Model, Disease - Free Equilibrium, Endemic Equilibrium, Stability, Basic Reproduction Number, Routh – Hurwitz Stability Criterion.*

Introduction

Tuberculosis disease is a highly contagious disease with person – to – person transmission mode, which attacks most of the population among the susceptible state. In most of the developing countries such as African countries due to poverty, lack of food safety and many other reasons [1]. This is one of the diseases, causes many deaths all over the world [2]. So, control and preventions are very important for human beings and also from economic point of view. For effective intervention we should know as much as possible for understanding of the disease transmission and burden they can impose in the population in many aspects. It threatens both children and adult.

Mathematical models are very useful tools in testing, comparing, planning, implementing the control intervention for this. [3] Shows that at the start of 20^{th} century, many researchers has been applying mathematical model for epidemic diseases. Some of the models are studied in [4], [5] and [6]. Deterministic models are those in which the population is divided into a finite number of homogeneous subpopulations. These models are formulating by using ordinary differential equations. These types of models provide the theoretical results like basic reproduction number, contact number etc. These types of models help to design an effective remedy to control the infectious disease [7]. There are many mathematical models have been studied for tuberculosis disease by sing deterministic model where the population is divided into subpopulations. SIR type model is discussed when one would not consider the exposed class. But in mathematical model for epidemic disease exposed class is very important one to see how the disease spread. If we use some preventive measures at the exposed class then we can control the spread of the disease at the starting phase of the disease. In African region, there is very less study having been done using the SEIR model to determine whether the Tuberculosis diseases will be epidemic or not [8]. The study shows that if it will get the favourable condition in future then it will be an epidemic which causes many deaths because of the sudden spread of the disease and due to the poverty of that African region; they cannot implement the prevention measures with great effect.

Here in this paper, we have discussed an SEIR model with birth and death at each compartment. We have also discussed the stability of the model by finding the basic reproduction number and using Routh – Hurwitz stability criterion.

Materials and Methods

The compartmental model that we discussed is an SEIR model when the total population is divided into subpopulation such as susceptible (S), Exposed (E), infectious (I) and Recovered (R). Also, the population is homogeneous that each of the susceptible population can get the disease when they are in contact with the infectious one. The SEIR model was originally developed by Kermack and McKendric [8]. Here we have calculated the basic reproduction number by using next generation matrix. Here we also have discussed the disease – free equilibrium and endemic equilibrium and test the stability of that equilibrium by using Routh – Hurwitz stability criterion.

If there are 8 deaths per 1000 population then the death rate (λ) = 0.008

and as we assumed that $\lambda = \mu = 0.008$.

Also, the recovery rate is reciprocal to the average period of infection so, if the average infectious period is 4 week the recovery rate is 0.25 per week.

And if the average latent period = 8 weeks

then the exposed rate (ϵ) = 0.125 per week.

The infection rate is the ratio of the effective contact over total contact.

Formation of the Model

Here we are going to discuss an SEIR model in which the total population is divided into four compartments i.e. S, E, I and R. Also, we have assumed that the total population is constant. We have considered here the birth rate which is same as the death rate. Here we have taken the death at each compartment.

We have used here the following notations -

λ= Birth rate

μ= Death rate

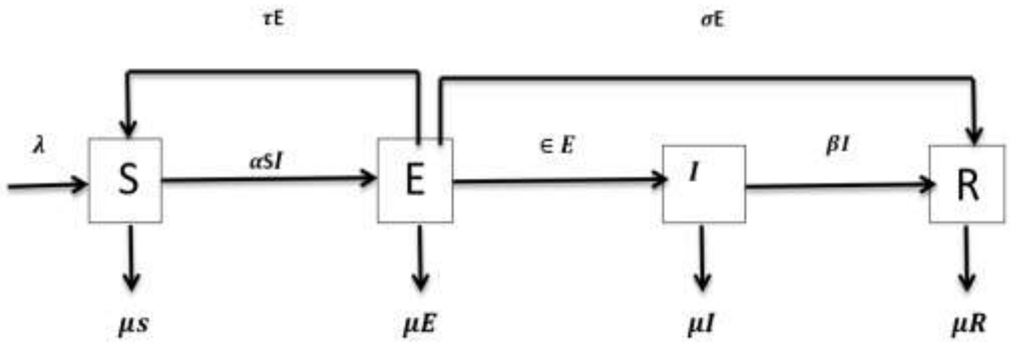
α= infectious rate

ϵ= the rate from which the population moves from the exposed class to infectious class

β= recovery rate

σ = rate at which the population moves from the exposed class to the recovered class after treatment.

τ= is the rate at which the population moves from the exposed class the susceptible class after treatment.



Result – 1

The ordinary differential equation for the above model is -

$\frac{dS}{dt} = \lambda - \mu S - \alpha SI + \tau E$

$\frac{dE}{dt} = \alpha SI - \mu E - \tau E - \sigma E - \epsilon E$ ……………………………….(1)

$\frac{dI}{dt} = \epsilon E - \mu I - \beta I$

$\frac{dR}{dt} = \beta I - \mu R + \sigma E$

Where $N = S + E + I + R$, where N represents the whole population.

Hence, $\frac{dN}{dt} = \frac{dS}{dt} + \frac{dE}{dt} + \frac{dI}{dt} + \frac{dR}{dt}$

$= \lambda - \mu(S + E + I + R)$

$= \lambda - \mu N$

$\Rightarrow \frac{dN}{dt} + \mu N = \lambda$

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This is a linear D.E.

Here, I. F. = $e^{\int \mu dt} = e^{\mu t}$

So, the solution of the above linear D.E. is -

$$N.e^{\mu t} = \int \lambda e^{\mu t} dt + k$$

$$= \frac{\lambda}{\mu} e^{\mu t} + k$$

N = $\frac{\lambda}{\mu} + ke^{-\mu t}$

At t = 0, N (0) = $\frac{\lambda}{\mu} + k$

k = N (0) - $\frac{\lambda}{\mu}$

so, $N(t) = N(0)e^{-\mu t} + \frac{\lambda}{\mu}(1 - e^{-\mu t})$

as, $t \to \infty, N(t) \leq \frac{\lambda}{\mu}$

Which shows that the solution set $\left\{(S, E, I, R): N = S + E + I + R \to \frac{\lambda}{\mu}\right\}$ is positively invariant set of the model.

Result – 2

Let S (0), E (0), I (0), R (0) ≥ 0, then the solution of the equation (1) is positive for all t ≥ 0.

Proof – We have, $\frac{dS}{dt} = \lambda - \mu S - \alpha SI + \tau E \geq -(\mu S + \alpha SI)$

$$\Rightarrow \frac{dS}{S} \geq -(\mu + \alpha I)dt$$

Integrating this we get,

$\log S(t) \geq -(\mu + \alpha I)t + c$

So, $S(t) \geq ce^{-(\mu + \alpha I)t}$

Initially at t = 0, we get from above,

$S(0) \geq c$

So, $S(t) \geq S(0)e^{-(\mu+\alpha I)t} \geq 0 \; since, S(0) \geq 0, e^{-(\mu+\alpha I)t} > 0,$

Hence, S(t) > 0.

Similarly, $\frac{dE}{dt} = \alpha SI - \mu E - \tau E - \sigma E - \epsilon E \geq -(\mu + \tau + \sigma + \epsilon)E$

$$\frac{dE}{E} \geq -(\mu + \tau + \sigma + \epsilon)dt$$

Integrating this we get,

$\log E(t) \geq -(\mu+\tau+\sigma+\epsilon)t + c$

$E(t) \geq ce^{-(\mu+\tau+\sigma+\epsilon)t}$

Initially, at t = 0, $E(0) \geq c$

$E(t) \geq E(0)e^{-(\mu+\tau+\sigma+\epsilon)t} \geq 0$, since $E(0) \geq 0, e^{-(\mu+\tau+\sigma+\epsilon)t} > 0$

So, E(t) > 0.

Similarly, $\frac{dI}{dt} = \epsilon E - \mu I - \beta I \geq -(\mu+\beta)I$

So, $\frac{dI}{I} \geq -(\mu+\beta)dt$

Integrating this we get,

$\log I(t) \geq -(\mu+\beta)t + c$

Hence, $I(t) \geq ce^{-(\mu+\beta)t}$

At, t = 0, $I(0) \geq c$

$I(t) \geq I(0)e^{-(\mu+\beta)t} \geq 0$

Therefore, $I(t) > 0$

Finally, $\frac{dR}{dt} = \beta I - \mu R + \sigma E > \sigma E - \mu R$

This can be written as, $\frac{dR}{dt} + \mu R > \sigma E$

This is a linear D.E.,

So, I. F. = $e^{\int \mu dt} = e^{\mu t}$

So, solution of the above equation is,

$R(t)e^{\mu t} > \frac{\sigma E e^{\mu t}}{\mu} + c$

Hence, $R(t) > \frac{\sigma E}{\mu} + ce^{-\mu t}$

AT t = 0, $R(0) > \frac{\sigma E}{\mu} + c$

$R(t) > R(0)e^{\mu t} + \frac{\sigma E}{\mu}(1 - e^{\mu t})$

So, $R(t) > 0$.

Hence, all variables are positive.

Steady State of D.F.E.

Here, $\frac{dS}{dt} = \frac{dE}{dt} = \frac{dI}{dt} = \frac{dR}{dt} = 0$

So,$\lambda - \mu S - \alpha SI + \tau E = 0$ …………………..(a)

$\alpha SI - \mu E - \tau E - \sigma E - \epsilon E = 0$ ……………(b)

$\epsilon E - \mu I - \beta I = 0$ ………………………………….(c)

$\beta I - \mu R + \sigma E = 0$ ………………………………..(d)

Form (a), we get, $\lambda = \mu S + \alpha SI - \tau E$

At, DFE, $\alpha = 0 \Rightarrow \lambda = \mu S - \tau E$………………(e)

From (b), $\alpha SI - \mu E - \tau E - \sigma E - \epsilon E = 0$

Since, $\alpha = 0, (\mu + \tau + \sigma + \epsilon)E = 0$

So, E = 0.

And from (e), $\lambda = \mu S \Rightarrow S = \frac{\lambda}{\mu}$

From (c), we have, $\epsilon E - \mu I - \beta I = 0$

As, E = 0. We get. $I = 0$

From (d), $\beta I - \mu R + \sigma E = 0$

Since, E = 0, $I = 0$, then R = 0.

Thus, DFE is $(S, E, I, R) = \left(\frac{\lambda}{\mu}, 0,0,0\right) = (1,0,0,0)$ as, $\lambda = \mu$.

Calculation of the R_0 −

Here we calculate R_0 as follows –

$$A - B = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \mu + \tau + \sigma + \epsilon & 0 \\ -\epsilon & \mu + \beta \end{bmatrix}$$

Where, A = matrix of infection

And B = matrix of transmission.

So, $A = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} \mu + \tau + \sigma + \epsilon & 0 \\ -\epsilon & \mu + \beta \end{bmatrix}$

Here, $|B| = (\mu + \tau + \sigma + \epsilon)(\mu + \beta)$

And $B^{-1} = \frac{1}{(\mu+\tau+\sigma+\epsilon)(\mu+\beta)} \begin{bmatrix} \mu + \beta & 0 \\ \epsilon & \mu + \tau + \sigma + \epsilon \end{bmatrix}$

So, $AB^{-1} = \frac{1}{(\mu+\tau+\sigma+\epsilon)(\mu+\beta)}\begin{bmatrix}0 & \alpha\\0 & 0\end{bmatrix}\begin{bmatrix}\mu+\beta & 0\\ \epsilon & \mu+\tau+\sigma+\epsilon\end{bmatrix}$

$= \frac{1}{(\mu+\tau+\sigma+\epsilon)(\mu+\beta)}\begin{bmatrix}\alpha\epsilon & \alpha(\mu+\tau+\sigma+\epsilon)\\0 & 0\end{bmatrix}$

So, R_0 is the dominant eigenvalue of AB^{-1}.

Hence, $R_0 = \frac{\alpha\epsilon}{(\mu+\tau+\sigma+\epsilon)(\mu+\beta)}$.

Steady State of EE

$\epsilon E - \mu I - \beta I = 0$

$\Rightarrow (\mu+\beta)I = \epsilon E$

$\Rightarrow I = \frac{\epsilon E}{(\mu+\beta)}$ ....(f)

Again, as, $\alpha SI - \mu E - \tau E - \sigma E - \epsilon E = 0$

$\Rightarrow S = \frac{(\mu+\tau+\sigma+\epsilon)E}{\alpha I} = \frac{(\mu+\tau+\sigma+\epsilon)(\mu+\beta)}{\alpha\epsilon}$

Hence, $S^* = \frac{(\mu+\tau+\sigma+\epsilon)(\mu+\beta)}{\alpha\epsilon}$

Again, $\lambda - \mu S - \alpha SI + \tau E = 0$

$\Rightarrow \lambda + \tau E = \mu S^* + \alpha S^* I$

After simplification we get, $E^* = \frac{1}{(\mu+\sigma+\epsilon)}\left[\lambda - \frac{\mu(\mu+\tau+\sigma+\epsilon)(\mu+\beta)}{\alpha\epsilon}\right]$

Since, $R_0 = \frac{\alpha\epsilon}{(\mu+\tau+\sigma+\epsilon)(\mu+\beta)}$

Then, $S^* = \frac{1}{R_0}$,

$E^* = \frac{\mu}{(\mu+\sigma+\epsilon)}\left[1 - \frac{1}{R_0}\right]$

$I^* = \frac{\epsilon\mu}{(\mu+\beta)(\mu+\sigma+\epsilon)}\left[1 - \frac{1}{R_0}\right]$

Stability Analysis of DFE Point

Theorem – The DFE of system of equation is also asymptotically stable when $R_0 < 1$ and unstable when $R_0 > 1$.

Proof – Let $k_1 = \lambda - \mu S - \alpha SI + \tau E$

$k_2 = \alpha SI - \mu E - \tau E - \sigma E - \epsilon E$

$k_3 = \epsilon E - \mu I - \beta I$

Then Jacobian, J = $\begin{bmatrix} \partial k_1/_{\partial S} & \partial k_1/_{\partial E} & \partial k_1/_{\partial I} \\ \partial k_2/_{\partial S} & \partial k_2/_{\partial E} & \partial k_2/_{\partial I} \\ \partial k_3/_{\partial S} & \partial k_3/_{\partial E} & \partial k_3/_{\partial I} \end{bmatrix}$

The Jacobian for the DFE for $(S, E, I) = (1, 0, 0)$.

$$J = \begin{bmatrix} -\mu & \tau & -\alpha \\ 0 & -(\mu+\tau+\sigma+\epsilon) & \alpha \\ 0 & \epsilon & -(\mu+\beta) \end{bmatrix}$$

Then the characteristic equation is –

$$|J - \lambda I_3| = \begin{bmatrix} -\mu-\lambda & \tau & -\alpha \\ 0 & -(\mu+\tau+\sigma+\epsilon)-\lambda & \alpha \\ 0 & \epsilon & -(\mu+\beta)-\lambda \end{bmatrix} = 0$$

After simplification we get,

$$\lambda^3 + P\lambda^2 + Q\lambda + R = 0$$

Where, P = $(3\mu + \tau + \sigma + \epsilon + \beta)$

Q = $\{\mu(2\mu + \tau + \sigma + \epsilon + \beta) + (\mu + \tau + \sigma + \epsilon)(\mu + \beta) - \alpha\epsilon\}$

R = $(\mu(\mu + \tau + \sigma + \epsilon)(\mu + \beta) - \mu\alpha\epsilon)$

Hence, by Routh – Hurwitz stability criterion, if P > 0, Q > 0 and PQ – R > 0, then all the roots of $|J - \lambda I_3| = 0$ have negative real part. So, DFE point is stable.

Stability Analysis of EE Point

Theorem – 2

The EE point is asymptotically stable if and only if $R_0 > 1$ and unstable if $R_0 < 1$.

Proof –

At EE, Jacobian, J = $\begin{bmatrix} -\mu-\alpha I^* & \tau & -\alpha S^* \\ \alpha I^* & -(\mu+\tau+\sigma+\epsilon) & \alpha S^* \\ 0 & \epsilon & -(\mu+\beta) \end{bmatrix}$

Then the characteristic equation of EE is –

$$|J - \lambda I_3| = \begin{bmatrix} -\mu-\alpha I^*-\lambda & \tau & -\alpha S^* \\ \alpha I^* & -(\mu+\tau+\sigma+\epsilon)-\lambda & \alpha S^* \\ 0 & \epsilon & -(\mu+\beta)-\lambda \end{bmatrix} = 0$$

After simplification we get,

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0$$

Where, A = $\alpha I^* - \mu - \tau - \epsilon - \beta$

B = $\{(2\mu + \tau + \sigma + \epsilon + \beta) + (\mu + \tau + \sigma + \epsilon)(\mu + \beta) - S^*\alpha\epsilon - \alpha\tau I^*\}$

C = $(\mu + \alpha I^*)(\mu + \tau + \sigma + \epsilon)(\mu + \beta) - \alpha\tau I^*(\mu + \beta) + \alpha^2\epsilon S^* I^*$

Hence, by Routh – Hurwitz stability criterion, if A > 0, B > 0 and AB – C > 0, then all the roots of $|J - \lambda I_3| = 0$ have negative real part. So, EE point is stable.

Graphical Representation

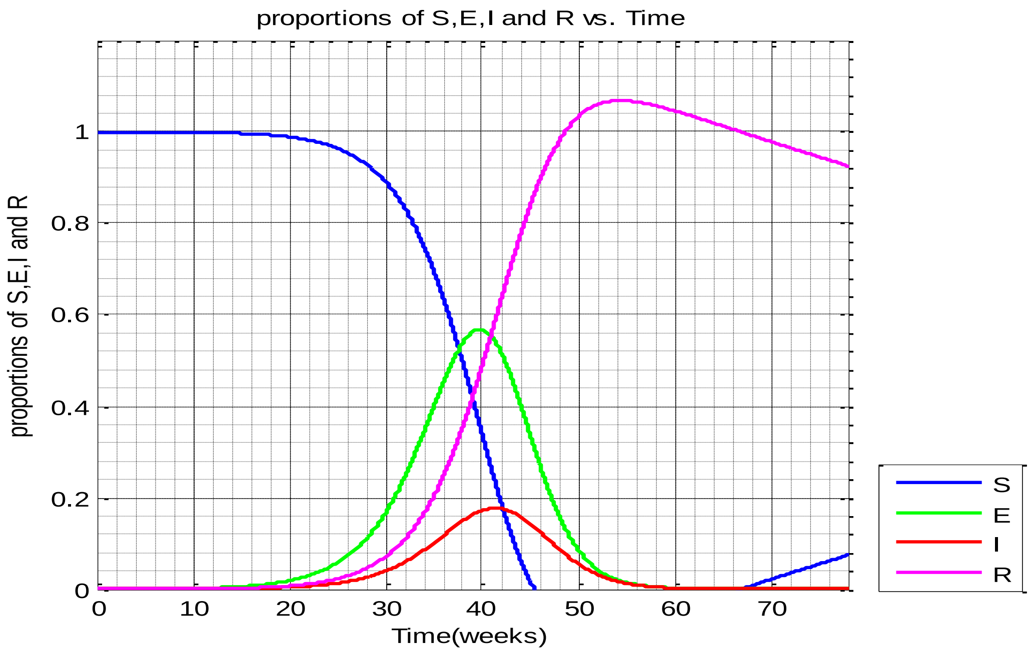
Here we are going to represent our SEIR model using some date by MATLAB Coding for some data given in the research articles.

Model Parameters and Sources –

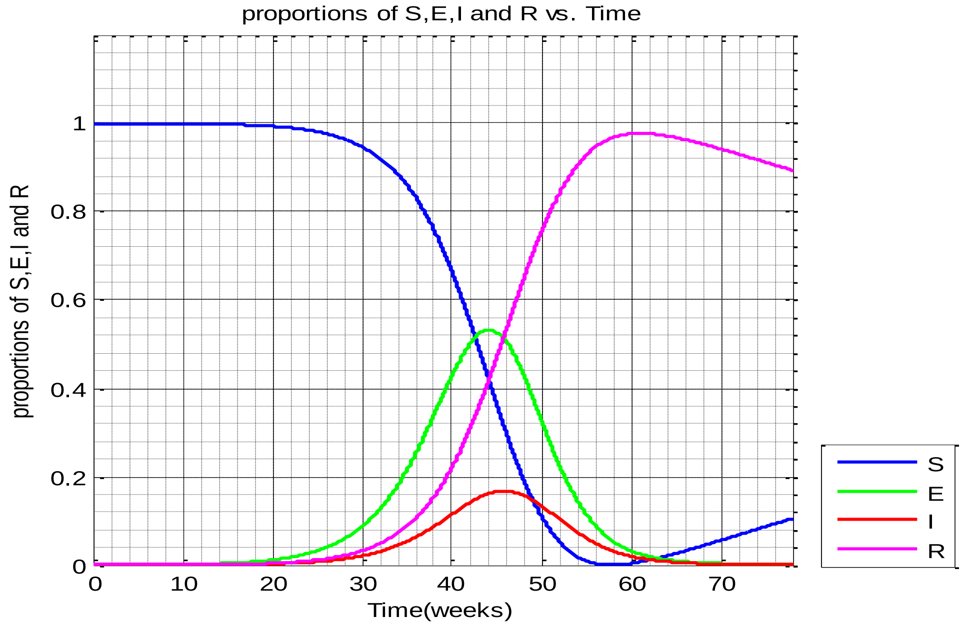
Parameters	Values	Sources
α	2	[9] and [10]
ϵ	0.1667	[9] and [10]
β	0.5	[11]
μ	0.007	[12]
λ	0.007	[12]
τ	0.02,0.04,0.06	Assumed
σ	0.02,0.04,0.06	Assumed

Here we have taken, birth rate (λ) = death rate (μ) and also assume that $\tau = \sigma$.

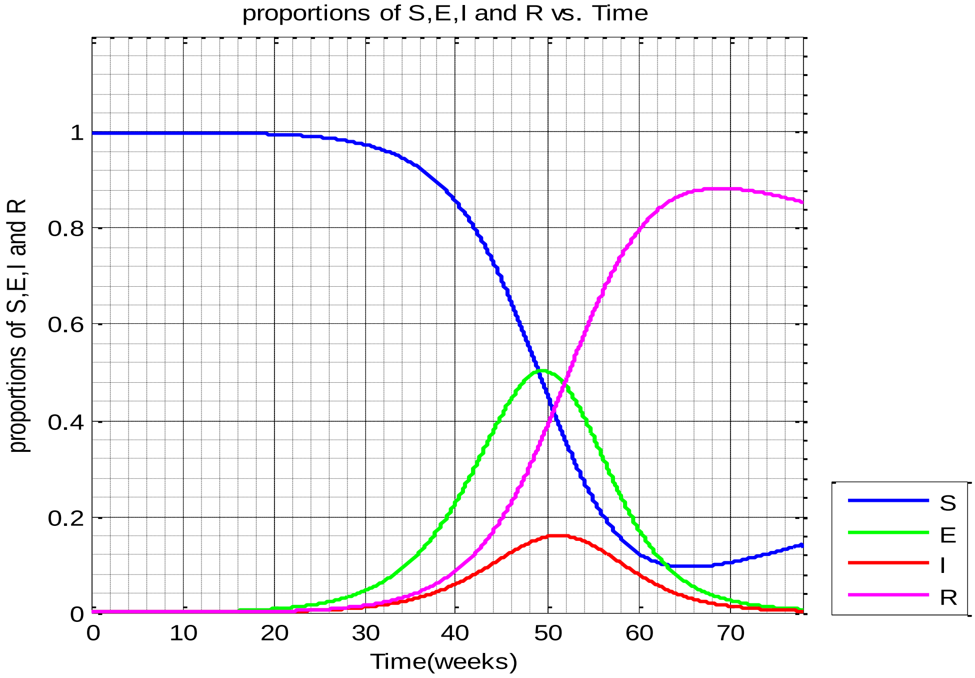
When $\tau = \sigma = 0.02$



When $\tau = \sigma = 0.04$



When $\tau = \sigma = 0.06$



From the above three graphs, we see that when the treatment at the latent class is applied for $\tau = \sigma = 0.02$, the time for maximum infectious population is at its peak at 41 weeks. Also, when $\tau = \sigma = 0.04$, it will be at 46 weeks and if $\tau = \sigma = 0.06$ then it will be 52 weeks. So latent class is very important in the epidemic model as it will give us some more time to think about it and work on it to decrease the rate of spread of the disease.

Conclusion

In this paper, we have discussed an SEIR compartmental model, in which the total population is divided into four subparts such as S, E, I and R. Here we have discussed that if we implement the prevention measures at the exposed class then we can make some effective policy that can reduce the speed of spread of the disease. And we get some more time to think and make some more important decisions. R_0 is very important to see how the disease spread into the population. It shows that the disease will spread very quickly, dies out after sometime or be there in the population.

Also, when we use the data given in some research articles, we see that when the treatment is imposed at the latent class when we get some more time to get the maximum infectious people and in that time we can implement some more prevention measures.

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